Iteration of quasiregular analogues of trigonometric functions

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September 2012

Joint work with Alastair Fletcher

Contents

- Introduction to quasiregular mappings of $\mathbb{R}^n$
- Dynamics of a quasiregular version of sine
- Dynamics of a quasiregular version of tangent

Quasiregular mappings

Quasiregular functions on $\mathbb{R}^n$ generalize analytic functions on $\mathbb{C}$.

Definition

- A continuous function $f : U \to \mathbb{R}^n$ on a domain $U \subseteq \mathbb{R}^n$ is called quasiregular if $f \in W^1_{n,\mathrm{loc}}(U)$ and there exists $K \geq 1$ such that
$$\|Df(\mathbf{x})\|^n \leq KJ_f(\mathbf{x}) \qquad \text{a.e. in } U.$$
- More generally, a continuous function $f : \mathbb{R}^n \to \mathbb{R}^n \cup \{\infty\}$ is called quasiregular (or quasimeromorphic) if the set of poles $f^{-1}(\infty)$ is discrete and if f is quasiregular on $\mathbb{R}^n \setminus f^{-1}(\infty)$.

The Zorich mapping

The Zorich map $Z : \mathbb{R}^3 \to \mathbb{R}^3 \setminus \{\mathbf{0}\}$ is a quasiregular analogue of the exponential function. It can be defined as follows:

1. Choose a bi-Lipschitz map
$$h : [-\tfrac{\pi}{2}, \tfrac{\pi}{2}]^2 \to \{(x, y, z) : x^2 + y^2 + z^2 = 1,\ z \geq 0\}.$$
2. Define $Z : [-\tfrac{\pi}{2}, \tfrac{\pi}{2}]^2 \times \mathbb{R} \to \{(x, y, z) : z \geq 0\}$ by
$$Z(x, y, z) = e^z h(x, y).$$
3. Extend Z to all of $\mathbb{R}^3$ by repeatedly reflecting in planes.

The Zorich map is quasiregular on $\mathbb{R}^3$ and doubly-periodic with periods $(2\pi, 0, 0)$ and $(0, 2\pi, 0)$.

Trigonometric analogues

- Quasiregular maps of $\mathbb{R}^n$ which generalize the sine and cosine functions have been constructed by Drasin, by Mayer and by Bergweiler and Eremenko.
- Constructed by mapping a *half*-infinite beam to a half-space, then reflecting in planes.
- By iterating their map S, Bergweiler and Eremenko obtained a seemingly paradoxical decomposition of $\mathbb{R}^n$.
- They also showed that the escaping set

$$I(S) = \{\mathbf{x} \in \mathbb{R}^n : S^k(\mathbf{x}) \to \infty \text{ as } k \to \infty\}$$

is dense in $\mathbb{R}^n$.

Dynamics of the qr sine analogue $S : \mathbb{R}^n \to \mathbb{R}^n$

We say $\mathbf{x}$ is a *periodic point* of S if $S^p(\mathbf{x}) = \mathbf{x}$ for some p.

Theorem

The periodic points of S are dense in $\mathbb{R}^n$.

Corollary

$\partial I(S) = \mathbb{R}^n$.

Theorem

S has the blowing-up property everywhere in $\mathbb{R}^n$; that is,

$$\bigcup_{k=0}^{\infty} S^k(U) = \mathbb{R}^n, \quad \textit{for any non-empty open } U \subseteq \mathbb{R}^n.$$

...so the “Julia set” of S is equal to $\mathbb{R}^n$.

In the rest of this talk we will

- construct a 3-dimensional quasimeromorphic analogue $T : \mathbb{R}^3 \to \mathbb{R}^3$ of the meromorphic tangent function

- compare the dynamics of λT and $\lambda \tan z$ for $\lambda > 0$.

Construction of a generalized tangent mapping

Observe that the complex function

$$\tan\zeta = \frac{i(1-e^{2i\zeta})}{1+e^{2i\zeta}}$$

is the composition of a Möbius map and the exponential function.

Define a sense-preserving Möbius map $A : \mathbb{R}^3 \to \mathbb{R}^3 \cup \{\infty\}$ by

$$A(x,y,z) = (0,0,1) + \frac{(2x, 2y, -2(z+1))}{x^2+y^2+(z+1)^2}.$$

We then define our 3-dimensional analogue of tangent by

$$T(\mathbf{x}) = (A \circ Z)(2\mathbf{x}).$$

Expressions for T

T contains embedded copies of the usual (complex) tangent function:

- $T(0, y, z) = (0,\ \mathrm{Re}(\tan(y + iz)),\ \mathrm{Im}(\tan(y + iz)))$,
- $T(x, 0, z) = (\mathrm{Re}(\tan(x + iz)),\ 0,\ \mathrm{Im}(\tan(x + iz)))$.

If $M(x, y) = \max\{|x|, |y|\} \le \pi/4$ and we write $\zeta = M(x, y) + iz$, then

$$T(x, y, z) = \left(\frac{x}{\sqrt{x^2 + y^2}} \mathrm{Re}(\tan \zeta), \frac{y}{\sqrt{x^2 + y^2}} \mathrm{Re}(\tan \zeta), \mathrm{Im}(\tan \zeta) \right).$$

Geometric properties of T

Comparing T with tan, the z-axis plays the role of the imaginary axis, while the xy-plane plays the role of the real axis.

- T is doubly-periodic with periods $(\pi, 0, 0)$ and $(0, \pi, 0)$.
- T omits the values $(0, 0, \pm 1)$. These are asymptotic values of T:

$$\lim_{z \to \pm\infty} T(x, y, z) = (0, 0, \pm 1).$$

- $T : \{xy\text{-plane}\} \to \{xy\text{-plane}\} \cup \{\infty\}$.
- The $\{z > 0\}$ and $\{z < 0\}$ half-spaces are completely invariant under T.

T is highly symmetric: If R is a reflection in a co-ordinate plane then

$$T(R(\mathbf{x})) = R(T(\mathbf{x})).$$

Iteration of tangent maps on $\mathbb{C}$

For a parameter $\lambda > 0$, Devaney and Keen described the dynamics of the meromorphic tangent family $\tau_\lambda(\zeta) = \lambda \tan \zeta$.

Theorem (Devaney and Keen)

- *If* $0 < \lambda < 1$, *then* $J(\tau_\lambda) \subseteq \mathbb{R}$ *is locally a Cantor set. Attracting fixed point at origin.*
- *If* $\lambda = 1$, *then* $J(\tau_\lambda) = \mathbb{R}$. *Parabolic fixed point at origin.*
- *If* $\lambda > 1$, *then* $J(\tau_\lambda) = \mathbb{R}$. *Attracting fixed points at* $\pm i\xi_0$, *where* $\xi_0 > 0$ *solves* $\xi_0 = \lambda \tanh \xi_0$.

Dynamics of λT

For $\lambda > 0$ we put

$$T_\lambda(\mathbf{x}) = \lambda T(\mathbf{x}).$$

We iterate T_λ and aim to establish an analogue of the $\lambda \tan \zeta$ results.

First, we describe the behaviour on the upper and lower half-spaces.

Theorem

- *If $0 < \lambda < 1$, then T_λ has an attracting fixed point at the origin.*
- *If $0 < \lambda \leq 1$, then $T_\lambda^k(\mathbf{x}) \to \mathbf{0}$ on $\{(x, y, z) : z \neq 0\}$, as $k \to \infty$.*
- *If $\lambda > 1$, then T_λ has attracting fixed points at $(0, 0, \pm\xi_0)$, where $\xi_0 = \lambda \tanh \xi_0$, and*

$$T_\lambda^k(\mathbf{x}) \to (0, 0, \pm\xi_0) \quad \text{on} \quad \{(x, y, z) : \pm z > 0\}.$$

What's a Julia set?

For a meromorphic function f with poles, the Julia set $J(f)$ satisfies

$$J(f) = \overline{O_f^-(\infty)} = \partial I(f),$$

where $I(f) = \{\zeta : f^k(\zeta) \to \infty \text{ as } k \to \infty\}$.

Theorem

For all $\lambda > 0$,

$$\overline{O_{T_\lambda}^-(\infty)} = \partial I(T_\lambda) = \overline{I(T_\lambda)}.$$

Call this set J. Then J is an uncountable perfect set.
If U is an open set that meets J then, for some $m > 0$,

$$T_\lambda^m(U) = (\mathbb{R}^3 \cup \{\infty\}) \setminus \{(0, 0, \pm\lambda)\}.$$

J is contained in the closure of the set of periodic points of T_λ.

What does J look like?

$$J = \overline{O^-_{T_\lambda}(\infty)} = \partial I(T_\lambda) \subseteq \{xy\text{-plane}\}$$

Theorem

If $\lambda \geq 1$ then J is connected. If $0 < \lambda < 1$ then J is not connected.

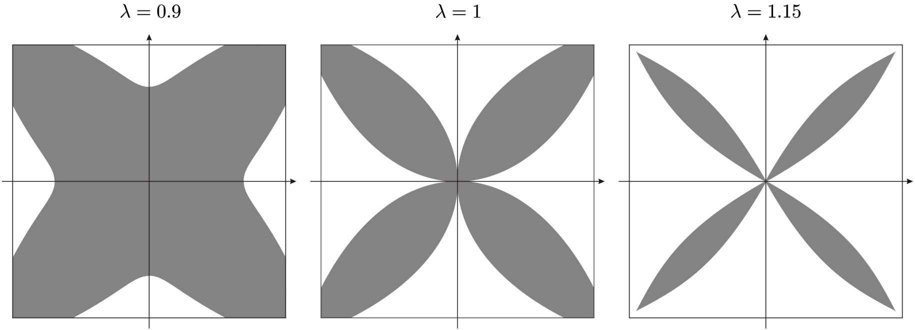
Open questions: When $0 < \lambda < 1$, is J locally Cantor?
Does J equal $\{\mathbf{x} : T^k_\lambda(\mathbf{x}) \not\to \mathbf{0}\}$?

Theorem

If $\lambda > \sqrt{2}$ then $J = \{xy\text{-plane}\}$.
The constant $\sqrt{2}$ here cannot be replaced by any smaller value.

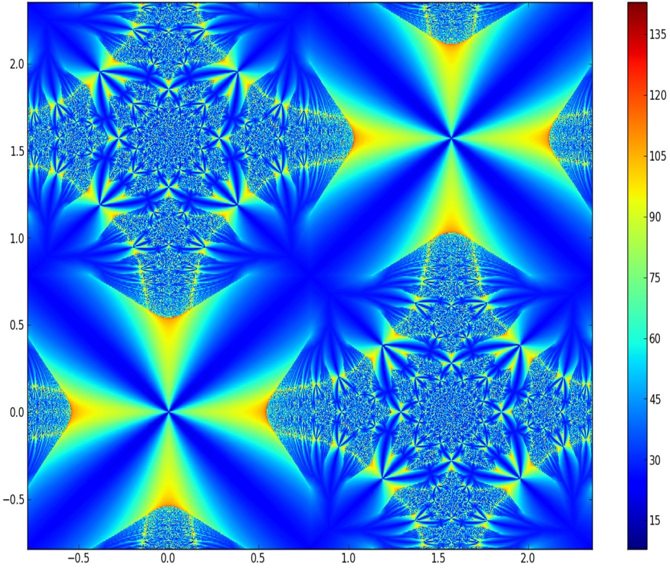
When $\lambda < \sqrt{2}$, a (relatively) open subset of the xy-plane lies in the attracting basin of $\mathbf{0}$ ...

Attracting basin of **0**

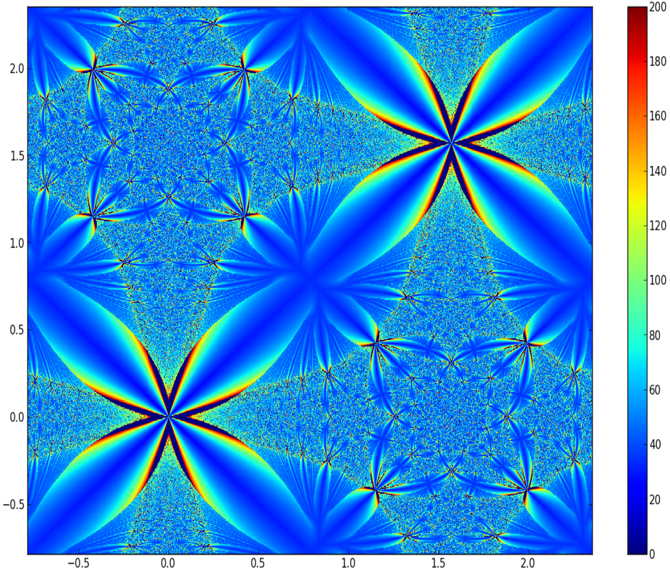


Each square is the subset $[-\frac{\pi}{4}, \frac{\pi}{4}]^2$ of the *xy*-plane.
The shaded points lie in the basin of attraction of **0**.

A numerical plot for $\lambda = 0.9$. Blue points $\rightarrow$ **0** fast, red points $\rightarrow$ **0** slow.



A numerical plot for $\lambda = 1$. Blue points $\to \mathbf{0}$ fast, red points $\to \mathbf{0}$ slow.



Around a pole for $\lambda = 0.7$. Thanks to Dan Goodman for code.